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EXAMINING THE SPECTRUM OF RADIATIVE TRANSFER SYSTEMS

Gladimir V. G. Baranoski
Department of Computer Science
University of Waterloo, Canada

Randall Bramley
Department of Computer Science
Indiana University, USA

Jon G. Rokne
Department of Computer Science
The University of Calgary, Canada

(Communicated by J.P. Hartnett and W.J. Minkowycz)

ABSTRACT

The convergence of iterative methods used to solve the linear systems of equations arising from discretizing radiative transfer problems depends on the characteristics of the eigenvalues of the coefficient matrix of these systems. In this communication we examine the proof that all of these eigenvalues are real and positive and discuss its implications for the solution of problems in radiative heat and mass transfer by iterative methods. This proof, outlined initially by Baranoski et al [1] in the context of the radiative transfer of luminous energy, allows the use of more efficient methods to solve radiative transfer systems. These methods, based on the knowledge about the set of eigenvalues of the radiative transfer coefficient matrix, may, in turn, provide faster solutions for these systems.

1 Introduction

In order to determine the fraction of diffusively radiated energy, or radiosity [2], from a surface of a given enclosure we need to solve the radiative transfer integral equation which discretized results in a system of linear equations [3]. We denote the coefficient matrix of this system by G. The vector of unknowns of this system is represented by the vector of radiosities b and the right-hand side vector of this system is represented by the vector of emittances e. This radiative transfer system is represented as:

$$Gb = e \tag{1}$$

where the elements of G are given by:

$$G_{ij} = \delta_{ij} - \rho_i F_{ij} \tag{2}$$

where:

δ_{il} = Kronecker delta,

ρ_i = reflectance of a surface i,

F_{ij} = form factor between a surface i and a surface j.

The form form factor F_{ij}, also called shape or view factor, indicates how a surface i ``sees" a surface j. In other words, it specifies the fraction of radiated energy that leaves a surface and i arrives at a surface j. Form factors depend on the shape and relative orientation of the surfaces as well as on the presence of obstacles between them. Several numerical algorithms have been developed to compute form factors. Generally speaking these methods can be divided into two groups: deterministic, based on quadrature methods [4], and nondeterministic, based on Monte Carlo methods [5]. The computational costs involved in the computation of form factors correspond in many cases to 90% of the total time spent to solve the radiative transfer integral equation [6]. These costs usually can be reduced through the application of the reciprocal rule for form factors [3]:

$$A_i F_{ij} = A_j F_{ji} \tag{3}$$

where:

A_i = area of surface i,

A_j = area of surface j.

Direct direct methods for solving linear systems, such as Gaussian Elimination [7] or LU decomposition [7] are not suitable for large radiative transfer systems of equations because of the relative low density of the coefficient matrix and because rapid solutions at relatively low accuracy are needed. These aspects plus the special properties of the radiative transfer coefficient matrix, namely its nonsingularity[1] and diagonal dominance[2] make the use of iterative methods more convenient.

Two questions come to mind when one investigates fast solutions for radiative transfer systems:

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[1] An $n \times n$ matrix K is said to be nonsingular if an $n \times n$ matrix K^{-1} (the inverse of K) exists such that $KK^{-1} = K^{-1}K = I$ [7]. Alternatively a matrix K is also said to be nonsingular if its determinant is nonzero [9].

[2] Assuming that $0 \le \rho_i < 1$ and that the sum of form factors in any row is equal to one, we can say that the matrix G is strictly diagonally dominant, since the property $|G_{ii}| > \sum_{j=1, j \neq i} |G_{ij}|$ holds for each $j = 1, \ldots, n$.

- Why is it relevant to search for faster iterative methods to solve the radiative transfer system of equations when the most expensive stage of the radiative transfer pipeline is the calculation of form factors?
- What if a radiative transfer system is too large to be stored in main memory of a computer system?

The answer for the first question involves applications in which a radiative transfer system has to be solved repeatedly, due to changes either to the vector of reflectances or to the vector of emittances, while the form factors remain unchanged. An example of such an application would be the simulation of the incidence of sunlight over a building during a day [8]. In this case the form factors would have to be computed only once, while the system would have to be solved 720 times to account for solar positions at every minute.

In order to address the second question one may resort to hierarchical approaches, which attempt to reduce the number of form factors that have to be computed and stored [10]. Applying these approaches the coefficient matrix becomes characterized by the presence of blocks along its diagonal. These diagonal blocks are radiative transfer subsystems with a manageable size that can be stored in main memory and even solved in parallel.

We believe that the aspects presented above justify the search for fast solutions for radiative transfer systems. In this communication we address issues directly related to this search. Section 2 outlines the relationship between the eigenvalues of the coefficient matrix of a radiative transfer system and the convergence of the iterative methods used to solve such a system. Section 3 examines the specific characteristics of these eigenvalues. The communication closes with considerations regarding the implications of the spectral characteristics of the radiative transfer coefficient matrix.

2 Relationship of Eigenvalues to Iterative Methods

An *eigenvector* v of a matrix G is a nonzero vector that does not rotate when G is applied to it, *i.e.* there is some scalar constant λ, an *eigenvalue* of G, such that $Gv = \lambda v$. Every square matrix G of order n has n possibly nondistinct complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. When G is symmetric the eigenvalues are real-valued. The set $\sigma(G)$ of eigenvalues of a matrix is called its *spectrum*.

Eigenvalues determine the convergence of iterative solvers used to solve linear systems such as $Gb = e$. For linear stationary methods of the form $b^{(j+1)} = Tb^{(j)} + \widetilde{e}$, which includes Jacobi, Gauss--Seidel, and SOR, the eigenvalues of the iteration matrix T are the relevant ones. The matrix T is derived from the coefficient matrix G ; for example in the Gauss-Seidel iteration $T = (D + L)^{-1}U$ where D, L, and U are the diagonal (D), strictly lower triangular, (L) and strictly upper (U) triangular parts of G. However, there might be no connection between the eigenvalues of T and those of G. Linear stationary methods converge if and only if $\zeta(T) < 1$, where $\zeta(T)$ is the size of the eigenvalue with largest magnitude. Furthermore, convergence is faster for smaller $\zeta(T)$.

For nonstationary methods such as Conjugate Gradient (CG) [11] or Chebyshev [12], the eigenvalues of the coefficient matrix G are the important ones. The Chebyshev method has convergence determined by the convex hull of the spectrum of G, which is determined by the extreme eigenvalues. For a matrix with positive real eigenvalues the largest ($\lambda_{\max}$) and smallest (λ_{mi}) eigenvalues completely determine convergence, which is faster for larger values of $(\lambda_{\max} + \lambda_{\min})/(\lambda_{\max} - \lambda_{\min})$. The convergence of the Conjugate Gradient method is determined by the overall distribution of eigenvalues, and even for a given $\sigma(G)$ it is impossible to predict the exact number of iterations CG will require to attain a given accuracy in the solution. However, CG generally requires only $s+1$ iterations for a given accuracy when the eigenvalues occur in only $s < n$ clusters, and the method has faster convergence for larger values of $(\lambda_{\max} + \lambda_{\min})/(\lambda_{\max} - \lambda_{\min})$.

In theory one could apply any iterative method to solve a radiative transfer system of linear of equations. However, if the eigenvalues of the coefficient matrix of this system are real and positive (Section 3), we can apply fast methods, like Chebyshev and Conjugate Gradient among others, with confidence of their convergence. In the next section we present the proof that these eigenvalues of the radiative transfer coefficient matrix are indeed real and positive.

3 Eigenvalues of the Radiative Transfer Coefficient Matrix

The transpose of an $n \times n$ matrix $G = (g_{ji})$ is the matrix $G^H = (g_{ij})$. A square matrix

G is said to be symmetric if $G = G^H$. Initially, to prove that all eigenvalues of the radiative transfer coefficient matrix G are real and positive, consider that it can be made symmetric by scaling its rows:

$$G^S = DG \tag{4}$$

where D is the diagonal matrix in which the diagonal entry d_{ii} is the quotient of the area and the reflectance of surface i. Recall the reciprocity relationship of form factors (Equation (3)).

Since DG is symmetric, its eigenvalues are real-valued. Moreover, by applying the Gerschgorin Circle Theorem [6], one can verify that they are also positive. Hence DG is a positive definite matrix [6]. The definition of positive definite means that:

$$x^H DGx > 0 \tag{5}$$

for all $x \in C$, where C is the complex plane and x^H is the Hermitian transpose of the vector x.

Let x be an eigenvector of G and λ be an eigenvalue. Then

$$Gx = \lambda x \tag{6}$$

where λ and $x \neq 0$ are possibly complex, which can be rewritten as:

$$DGx = \lambda Dx \tag{7}$$

and

$$x^H DGx = \lambda x^H Dx \tag{8}$$

The left side of Equation (8) is necessarily real and positive from Equation (5). Furthermore, the definition of an eigenvector x implies that it is nonzero, hence

$$x^H Dx = \sum \bar{x}_i x_i d_{ii} \geq d_{\min} \sum x_i \bar{x}_i = d_{\min} x^H x = d_{\min} \|x\| > 0 \tag{9}$$

where $d_{\min} = \min_i d_{ii}$.

Equations (8) and (9) imply that λ is real and $\lambda > 0$. Therefore all of the eigenvalues of G are real and positive.

4 Concluding Remarks

Although it may seem that this proof applies directly to the continuous radiative transfer operator, more details need to be considered. The critical point is that the diagonal entry i of the matrix D is the ratio of the area and reflectance of the i-th surface. For the continuous case, the area is zero and the above argument cannot be used directly. However, the continuous operator is a compact operator, sometimes also called a completely continuous operator in functional analysis [13]. Because of this we can construct a sequence of finite dimensional operators G_k that converge uniformly to the continuous operator G_∞, each with a spectrum consisting only of positive real eigenvalues. That sequence can be constructed using a sequence of uniformly refined discretizations of the enclosure, for example. The limit G_∞ will necessarily have a spectrum that is real and nonnegative. Because G_∞ is nonsingular, it cannot have zero as an eigenvalue. Its compactness also implies that it has only a point spectrum, which implies that G_∞ has only positive real eigenvalues in its spectrum.

Finally, we would like to stress the significance of this proof for the solution of problems involving the transport of radiant energy. As examined in Section 2, the fact that all eigenvalues of G are real and positive allows the application of fast iterative solvers, such as Chebyshev and Conjugate Gradient, to radiative transfer systems with confidence of their convergence.

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